MATH 121A Prep: Bases

Facts to Know:

$A = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_m} \end{bmatrix}$

Let $\vec{v_1}, \ldots, \vec{v_m}$ be vectors in a subspace $V \subset \mathbb{R}^n$, and A the $n \times m$ matrix with columns $\vec{v_1}, \ldots, \vec{v_m}$.

$\vec{v_1}, \ldots, \vec{v_m}$ are Linearly Independent if: $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_m\vec{v_m} = \vec{0}$ then $c_1 = c_2 = \cdots = 0$.

$\iff A\vec{c} = \vec{0}$ having only the trivial solution $(\vec{c} = \vec{0})$

$\vec{v_1}, \ldots, \vec{v_m}$ Span V if: $\forall \vec{w} \in V$ there are $c_1, \ldots, c_m$ s.t. $\vec{w} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_m\vec{v_m}$

$\iff A\vec{c} = \vec{w}$ has a solution

$\vec{v_1}, \ldots, \vec{v_m}$ is a Basis for V if: they are linearly independent and they span V.

Dimension of a Subspace V: (1) Find basis $\vec{v_1}, \ldots, \vec{v_m}$

(2) Dimension of $V = m$ = # of vectors in a basis for V

Examples:

1. Determine whether the vectors $\vec{v_1} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ are linearly independent.

$A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -1 & 3 \\ 2 & 1 & 0 \end{bmatrix}$ WTS $A\vec{c} = \vec{0}$ has only trivial solution.

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 1 & -1 & 3 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R1 \leftrightarrow R2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R3 = R3 - 2R1]{R2 = R2 - 2R1} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 3 & -6 & 0 \end{array}\right] \xrightarrow{R2 = \frac{1}{2}R2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & -6 & 0 \end{array}\right]$$

$\xrightarrow{R3 = R3 - 3R2}$ $\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ x_3 is free variable $\rightarrow$ infinitely many solutions

NOT linearly independent.

2. (a) Show that the vectors $\vec{v_1} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ span $\mathbb{R}^2$. Write $\vec{w} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$ as a linear combination of $\vec{v_1}$ and $\vec{v_2}$.

$A = \begin{bmatrix} 1 & -2 \\ 4 & 1 \end{bmatrix}$ show $A\vec{c} = \vec{b}$ always has a solution

$\left[\begin{array}{cc|c} 1 & -2 & b_1 \\ 4 & 1 & b_2 \end{array}\right] \xrightarrow{R2 = R2 - 4R1} \left[\begin{array}{cc|c} 1 & -2 & b_1 \\ 0 & 9 & b_2 - 4b_1 \end{array}\right]$

$\xrightarrow{R2 = \frac{1}{9}R2} \left[\begin{array}{cc|c} 1 & -2 & b_1 \\ 0 & 1 & \frac{b_2 - 4b_1}{9} \end{array}\right]$ leading 1 in each row means we can always find solution!

So spans $\mathbb{R}^2$

$A\vec{c} = \vec{w}$ $\left[\begin{array}{cc|c} 1 & -2 & -5 \\ 4 & 1 & 7 \end{array}\right] \xrightarrow{R2 = R2 - 4R1} \left[\begin{array}{cc|c} 1 & -2 & -5 \\ 0 & 9 & 27 \end{array}\right]$ $\boxed{\vec{w} = \vec{v_1} + 3\vec{v_2}}$

$\xrightarrow{R2 = \frac{1}{9}R2} \left[\begin{array}{cc|c} 1 & -2 & -5 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{R1 = R1 + 2R2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \end{array}\right]$ $c_1 = 1$, $c_2 = 3$

(b) Show that $\vec{v_1}$ and $\vec{v_2}$ are also linearly independent so they form a basis for $\mathbb{R}^2$.

* $A\vec{c} = \vec{0}$ has only trivial solution

$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 4 & 1 & 0 \end{array}\right] \xrightarrow[\text{operations as (a)}]{\text{same row}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \rightarrow \begin{array}{l} c_1 = 0 \\ c_2 = 0 \end{array}$

So $\vec{v_1}, \vec{v_2}$ are linearly independent.

Linearly independent + span $\mathbb{R}^2 \rightarrow$ basis for $\mathbb{R}^2$

($\mathbb{R}^2$ has dimension 2)

3. Can 2 vectors span $\mathbb{R}^3$? $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^3$ that span $\mathbb{R}^3$? No.

$$\text{3 rows} \left[\begin{array}{cc|c} \vec{v}_1 & \vec{v}_2 & \vec{w} \end{array} \right] \longleftrightarrow A\vec{c} = \vec{w} \quad A = [\vec{v}_1 \ \vec{v}_2]$$

2 cols

at best: $\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{array} \right]$ I can always choose $\vec{w}$ so that there are non-zero entries in the last column.

$* =$ non-zero value ✗ no solution

* You need at least n vectors to span $\mathbb{R}^n$

4. Can 3 vectors be linearly independent in $\mathbb{R}^2$? $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^2$ No.

$$A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$$

$A\vec{c} = \vec{0}$ can't have trivial soln.

$$\text{2 rows} \left[\begin{array}{ccc|c} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{0} \end{array} \right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 0 & \emptyset & 0 \\ 0 & 1 & \emptyset & 0 \end{array} \right]$$

column w/out leading 1

Extra column → Free variable

→ infinitely many solutions

→ not linearly independent.

* At **most** n vectors can be linearly independent in $\mathbb{R}^n$.